

Gibbs properties of self-conformal measures and the multifractal formalism

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Abstract. We prove that for any self-conformal measures, without any separation conditions, the multifractal formalism partially holds. The result follows by establishing certain Gibbs properties for self-conformal measures.

1. Introduction

This paper is devoted to the study of the multifractal structure of self-conformal measures. To state our the results, let us first give some notations and backgrounds.

One of the main objectives of multifractal analysis is to study the dimension spectra and their relation with the L^q spectra for a given measure. Let ν be a compactly supported Borel probability measure on \mathbb{R}^d . For $x \in \mathbb{R}^d$, the *upper and lower local dimensions* of ν at x are defined by

$$\bar{d}(\nu, x) = \limsup_{r \rightarrow 0^+} \frac{\log \nu(B(x, r))}{\log r}, \quad \underline{d}(\nu, x) = \liminf_{r \rightarrow 0^+} \frac{\log \nu(B(x, r))}{\log r},$$

where $B(x, r)$ stands for the closed ball of radius r centered at x . When $\bar{d}(\nu, x) = \underline{d}(\nu, x)$, the common value, denoted by $d(\nu, x)$, is called the *local dimension* of ν at x . For $\alpha \geq 0$, define

$$E_\nu(\alpha) = \{x \in \mathbb{R}^d : d(\nu, x) = \alpha\}, \quad f_\nu(\alpha) = \dim_H E_\nu(\alpha),$$

where \dim_H denotes the Hausdorff dimension (see, e.g., [13] for a definition). The $E_\nu(\alpha)$ are called *the level sets* of ν , and $f_\nu(\alpha)$ the *dimension spectra* of ν . For each $n \geq 1$, let \mathbf{D}_n be the set of cubes $\{[0, 2^{-n}]^d + \alpha : \alpha \in 2^{-n}\mathbb{Z}^d\}$. For $q > 0$, define

$$\tau_n(\nu, q) = \sum_{Q \in \mathbf{D}_n} (\nu(Q))^q. \tag{1}$$

The L^q spectrum of ν is defined as

$$\tau(\nu, q) = \lim_{n \rightarrow \infty} \frac{\log \tau_n(\nu, q)}{-n \log 2}, \quad (2)$$

if the limit exists. In general, we use $\underline{\tau}(\nu, q)$ to denote the corresponding value by taking the lower limit, and call it the *lower L^q spectrum* of ν . Moreover $\underline{\tau}(\nu, q)$ can be defined over \mathbb{R} by

$$\underline{\tau}(\nu, q) = \liminf_{r \rightarrow 0^+} \frac{\log(\sup \sum_i \nu(B(x_i, r))^q)}{\log r},$$

where the supremum is taken over all the disjoint families $\{B(x_i, r)\}_i$ of closed balls with $x_i \in \text{supp}(\nu)$. It is easily checked that $\underline{\tau}(\nu, q)$ is a concave function of q over \mathbb{R} .

The self-similar measures and self-conformal measures are typical multifractal measures. To introduce some corresponding notations, let $U \subset \mathbb{R}^d$ be an open set. A C^1 -map $S : U \rightarrow \mathbb{R}^d$ is *conformal* if the differential $S'(x) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies $|S'(x)y| = |S'(x)| \cdot |y| \neq 0$ for all $x \in U$ and $y \in \mathbb{R}^d$, $y \neq 0$. Furthermore, $S : U \rightarrow \mathbb{R}^d$ is *contracting* if there exists $0 < \gamma < 1$ such that $|S(x) - S(y)| \leq \gamma \cdot |x - y|$ for all $x, y \in U$. We say that $\{S_i : X \rightarrow X\}_{i=1}^\ell$ is a *C^1 -conformal iterated function system* (C^1 -conformal IFS) on a compact set $X \subset \mathbb{R}^d$ if each S_i extends to an injective contracting C^1 -conformal map $S_i : U \rightarrow U$ on an open set $U \supset X$. Let $\{S_i\}_{i=1}^\ell$ be a C^1 -conformal IFS on a compact set $X \subset \mathbb{R}^d$. It is well-known, see [29], that there is a unique non-empty compact set $K \subset X$ such that $K = \bigcup_{i=1}^\ell S_i(K)$. Given a probability vector (p_1, \dots, p_ℓ) , there is a unique Borel probability measure ν satisfying

$$\nu = \sum_{i=1}^\ell p_i \nu \circ S_i^{-1}. \quad (3)$$

This measure is supported on K and it is called *self-conformal*. In particular, if the maps S_i are all similitudes, then ν is called *self-similar*.

We point out that for any self-conformal measure ν on \mathbb{R}^d , the limit $\tau(\nu, q)$ in (2) always exists for any $q > 0$. This fact was first proved by Peres and Solomyak [44] under an additional assumption that the generating IFS $\{S_i\}_{i=1}^\ell$ for ν satisfies the *bounded distortion property*: There exists $L \geq 1$ such that for every $n \in \mathbb{N}$ and for every word $u = u_1 \dots u_n \in \{1, \dots, \ell\}^n$,

$$L^{-1} \leq \frac{\|S'_u(x)\|}{\|S'_u(y)\|} \leq L, \quad \forall x, y \in U, \quad (4)$$

where $S_u = S_{u_1} \circ \dots \circ S_{u_n}$. We will show that this assumption can be removed (see Corollary 4.5), and furthermore $\tau(\nu, q)$ always takes values in \mathbb{R} (see Lemma 2.4).

For a given measure, usually it is very hard or impossible to calculate the corresponding dimension spectra directly. The celebrated heuristic principle known as the *multifractal formalism*, which was first introduced by some physicists [22, 25, 26], states that the dimension spectra $f_\nu(\alpha)$ and the lower L^q -spectra $\underline{\tau}(\nu, q)$ form a Legendre-transform pair, i.e.,

$$f_\nu(\alpha) = \tau^*(\alpha) := \inf\{\alpha q - \underline{\tau}(\nu, q) : q \in \mathbb{R}\}.$$

For rigorous mathematical foundations of the multifractal formalism, we refer to [6, 7, 41, 47, 49]. Although false in general, the multifractal formalism has been verified for many natural measures, including, *e.g.*, Gibbs measures of conformal dynamical systems (discrete or continuous time) [24, 46, 4, 45], cookie-cutter Cantor measures [9, 51], quasi-Bernoulli measures [7, 27], weak Gibbs measures [21, 58]. Here for Gibbs measures of conformal dynamical systems, there is no separation condition required for the systems. For such kind of systems, for example, conformal repellers, one can obtain their symbolic representations via Markov partitions, and then apply the classical thermodynamical formalism to construct equilibrium measures supported on $E_\nu(\alpha)$ to estimate its dimension directly (see [46] for details). In most of the literature, an additional bounded distortion property similar to (4) is required for the systems to guarantee the ergodicity and Gibbs property of equilibrium measures, as well as the differentiability of L^q spectra. In a recent work [3], Barreira and Gelfert considered the multifractal structure of Lyapunov exponents on non-conformal repellers and obtained some related results without requiring the bounded distortion property.

We point out that the multifractal formalism has also been verified for some typical fractal measures, for instance, self-similar measures and self-conformal measures under some separation assumptions. In [8], Cawley and Mauldin verified the multifractal formalism for self-similar measures under the strong separation condition, in which $\tau(\nu, q)$ is given explicitly by

$$\sum_{i=1}^{\ell} p_i^q \rho_i^{-\tau(\nu, q)} = 1,$$

where ρ_i is the contraction ratio of the similitude S_i . An extension was given to graph-directed constructions of measures by Edgar and Mauldin [11]. Later Patzschke [43] and Fan and Lau [14] extended the result to self-conformal measures satisfying the bounded distortion property and the well-known open set condition (see [29]), where $\tau(\nu, q)$ is proved to be an analytic function over \mathbb{R} . Under the same conditions, the self-conformal measures generated by countably many contractive conformal maps were considered by Riedi and Mandelbrot [52], Mauldin and Urbański [37], Pollicott and Weiss [50]. The random self-similar measures with the open set condition were considered in [35, 30, 12, 42, 1, 2]. In [32], Lau and Ngai introduced a notion “weak separation condition” (WSC) which is weaker than the open set condition and includes many interesting overlapping IFS, such as the Bernoulli convolutions associated with Pisot numbers (see, *e.g.*, [16] for details). They proved that the multifractal formalism still partially holds for self-similar measures under the WSC. In recent years there have been a lot of interest in the multifractal analysis for this kind of overlapping self-similar measures, and many exceptional multifractal phenomena have been found (see, *e.g.*, [34, 31, 28, 15, 16, 17, 19, 21, 33, 20, 40, 54, 55, 56, 10, 57]).

It is a natural question whether or not the multifractal formalism still holds or partially holds for any self-similar measures and self-conformal measures *without any separation conditions*, i.e., whether or not the multifractal formalism is

generically (partially) valid for measures with self-similar properties. To our best knowledge, so far there have been no known results for this question. The main difficulty lies in the fact that there is no known effective method to analyze the local fine structures for such overlapping measures. As a result, one can not expect to use directly the thermodynamic formalism in dynamical systems or the large deviation theory to construct a one-parameter family of equilibrium measures (corresponding to a real or matrix-valued continuous function) supported on $E_\nu(\alpha)$ having the full dimension in the classic way. Nevertheless, in the present paper we obtain the first result for the question.

THEOREM 1.1. *Let ν be a self-conformal measure on \mathbb{R}^d , without any separation conditions. Then for any $\alpha = \tau'(\nu, t)$ with $t \geq 1$ (provided that $\tau(\nu, \cdot)$ is differentiable at t),*

$$\dim_H E_\nu(\alpha) = \alpha t - \tau(\nu, t) = \inf\{\alpha q - \underline{\tau}(\nu, q) : q \in \mathbb{R}\}. \quad (5)$$

Since $\tau(\nu, q)$ is finite and concave on $(0, \infty)$, it is differentiable except for at most countably many points $q \in (0, \infty)$. Hence there do exist $\alpha \geq 0$ such that the multifractal formula (5) holds. The proof of Theorem 1.1 is based on the following Gibbs properties of self-conformal measures.

THEOREM 1.2. *Let ν be an arbitrary self-conformal measure on \mathbb{R}^d . Then for any $q > 0$, there exist a Borel probability measure ν_q on \mathbb{R}^d and a positive function $h(r)$ (which depends q) with $\lim_{r \rightarrow 0} \frac{\log h(r)}{\log(1/r)} = 0$ such that for $q \geq 1$,*

$$\nu_q \left(B(x, r/(16\sqrt{d})) \right) \leq h(r) r^{-\tau(\nu, q)} \nu(B(x, r))^q, \quad \forall x \in \mathbb{R}^d, 0 < r < 1, \quad (6)$$

and for $0 < q < 1$,

$$\nu_q \left(B(x, 16\sqrt{d}r) \right) \geq h(r) r^{-\tau(\nu, q)} \nu(B(x, r))^q, \quad \forall x \in \mathbb{R}^d, 0 < r < 1/4. \quad (7)$$

In fact, Theorem 1.1 follows directly from the Gibbs property (6) and a multifractal result about measures (see Proposition 2.1). We remark that in Theorem 1.2 the positive function $h(r)$ can be replaced by a positive constant if the generating IFS for ν satisfies the bounded distortion property (4).

We point out that Theorem 1.1 and Theorem 1.2 can be extended to a broader class of probability measures supported on self-conformal sets. To be more precisely, let $\{S_i\}_{i=1}^\ell$ be a conformal IFS in \mathbb{R}^d and let K be the corresponding self-conformal set. We use $\Sigma = \{1, \dots, \ell\}^{\mathbb{N}}$ to denote the one-sided full shift space over the alphabet $\{1, \dots, \ell\}$. Consider the canonical projection $\pi : \Sigma \rightarrow K$ defined by

$$\pi(x) = \lim_{n \rightarrow \infty} S_{x_1} \circ \dots \circ S_{x_n}(0), \quad x = (x_i)_{i=1}^\infty \in \Sigma. \quad (8)$$

Assume that μ is a Borel probability measure on Σ such that for some constant $C > 0$,

$$\mu([IJ]) \leq C \mu([I]) \mu([J]) \quad (9)$$

for any two words $I = i_1 \dots i_n$ and $J = j_1 \dots j_m$ over $\{1, \dots, \ell\}$, where $[u_1 \dots u_n]$ denotes the cylinder set $\{x = (x_i)_{i=1}^\infty \in \Sigma : x_i = u_i \text{ for } 1 \leq i \leq n\}$. Then the

measure $\nu = \mu \circ \pi^{-1}$, called *the projection of μ under π* , has the Gibbs property (6); and Theorem 1.1 holds for this measure. Accordingly, if μ is a Borel probability measure on Σ such that for some constant $C' > 0$,

$$\mu([IJ]) \geq C' \mu([I])\mu([J]) \quad (10)$$

for any two finite words I and J over $\{1, \dots, \ell\}$, then the corresponding measure $\nu = \mu \circ \pi^{-1}$ has the Gibbs property (7). For some details, see Remark 3.10 and Theorem 4.6.

We do not know whether or not the result of Theorem 1.1 can be extended to all $t > 0$ for general self-conformal measures. However this is true for a class of self-conformal measures, for which the generating IFS satisfy the so-called *asymptotically weak separation condition* (see Definition 5.1 and Theorem 5.7). For example, consider the simplest family of IFS $\Phi_\rho = \{\rho x, \rho x + 1\}$, where $0 < \rho < 1$. Then Φ_ρ satisfies the asymptotically weak separation condition when $1/\rho$ is a Pisot number or Salem number (see Definition 5.2 and Proposition 5.3).

An interesting question arises that whether or not $\tau(\nu, q)$ is always a differentiable function of q on $(0, \infty)$ for a general self-similar or self-conformal measure ν . We conjecture that it is true, at least for self-similar measures. For some self-similar measures satisfying certain separation condition, this was proved to be true. To be more precisely, let $\{S_i(x) = \rho x + d_i\}_{i=1}^\ell$ be an IFS on \mathbb{R} consisting of equi-contractive similitudes, and K the corresponding self-similar set. Following Ngai and Wang [39], say $\{S_i\}_{i=1}^\ell$ satisfies the *finite type condition* if the set

$$\bigcup_{n=1}^{\infty} \{\rho^{-n}(S_u(0) - S_v(0)) : u, v \in \{1, \dots, \ell\}^n\} \cap [0, \text{diam}(K)]$$

is finite. For example, the IFS $\{\rho x + d_i\}_{i=1}^\ell$ satisfies the finite type condition if $1/\rho$ is a Pisot number and all d_i are integers (see, *e.g.* [32, 39]). The author showed in [15] that if the IFS $\{\rho x + d_i\}_{i=1}^\ell$ satisfies the finite type condition, then $\tau(\nu, q)$ is differentiable over $(0, \infty)$ for the corresponding self-similar measure ν . The result is based on the thermodynamic formalism for matrix-valued functions established in [19]. Some extensions were given recently in [55, 10] for some specific non-equi-contractive and high dimensional cases. We remark that even under the finite type condition, $\tau(\nu, q)$ may be non-differentiable for some $q < 0$ and it may lead to intervals in which the multifractal formalism does not hold (see [16, 21, 20, 28, 33, 54, 55, 56]). In particular, Testud constructed some simple class of self-similar measures on \mathbb{R} of which the dimension spectra are very wide and not concave [56].

By the way, we point out that any self-conformal measure ν should be exact dimensional, *i.e.*, the local dimensions $d(\nu, x)$ are equal to a constant for ν -a.e. x . The statement was claimed a long time ago by Ledrappier, whilst a rigorous proof of this fact was carried out in [18] by the author and Hu recently.

The paper is arranged in the following manner: in section 2, we present a multifractal result for measures satisfying certain Gibbs property; in section 3 we setup some inequalities for self-conformal measures; in section 4 we verify the Gibbs

properties for these measures, and prove Theorem 1.2 and Theorem 1.1; in section 5, we study C^1 -conformal IFSs which satisfy the asymptotically weak separation condition.

2. A multifractal result for measures satisfying certain Gibbs property

In this section, we provide the following proposition, which extends a result of Ben Nasr [5] about measures μ on full shift spaces satisfying (9). For the convenience of the readers, we give a complete proof, the idea of which is essentially due to Brown, Michon and Peyrière [7].

PROPOSITION 2.1. *Let ν be a compactly supported Borel probability measure on \mathbb{R}^d , and (a, b) an interval with $a > 0$. Assume that the L^q spectrum $\tau(q) := \tau(\nu, q)$ of ν exists on (a, b) . Furthermore assume that for any $q \in (a, b)$, there exist a constant $t > 1$ which is independent of r , a map $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\lim_{r \rightarrow 0} \frac{\log h(r)}{\log r} = 0$ and a Borel probability measure ν_q on \mathbb{R}^d and $r_0 > 0$ such that*

$$\nu_q(B(x, t^{-1}r)) \leq h(r) \nu(B(x, r))^q r^{-\tau(q)}, \quad \forall x \in \mathbb{R}^d, 0 < r < r_0. \quad (11)$$

Then for any $q \in (a, b)$ and ν_q almost all $x \in \mathbb{R}^d$,

$$\underline{d}(\nu, x) \geq \tau'(q+), \quad \bar{d}(\nu, x) \leq \tau'(q-). \quad (12)$$

In particular when τ is differentiable at q , we have $d(\nu, x) = \tau'(q)$ for ν_q almost all $x \in \mathbb{R}^d$, and furthermore

$$\dim_H E_\nu(\alpha) = \tau^*(\alpha) = \alpha q - \tau(q). \quad (13)$$

for $\alpha = \tau'(q)$.

The above proposition is also related to a general result in [6]. To prove the proposition, we need some lemmas. First we recall the following covering lemma. For a proof one is referred to Mattila's book [36, p. 30].

LEMMA 2.2. (*Besicovitch's covering lemma*). *Let A be a bounded subset of \mathbb{R}^d , and let \mathcal{B} be a family of closed balls such that each point of A is the center of some ball of \mathcal{B} . Then there are families $\mathcal{B}_1, \dots, \mathcal{B}_c \subset \mathcal{B}$ covering A , where c is a constant only depending on d , such that each \mathcal{B}_i is disjoint, that is,*

$$A \subset \bigcup_{i=1}^c (\cup \mathcal{B}_i)$$

and $B \cap B' = \emptyset$ for $B, B' \in \mathcal{B}_i$ with $B \neq B'$, where $\cup \mathcal{B}_i = \bigcup_{B \in \mathcal{B}_i} B$.

The following lemma is obvious.

LEMMA 2.3. *Let $q > 0$. For any $k \in \mathbb{N}$ and non-negative numbers x_1, \dots, x_k ,*

$$\frac{1}{k}(x_1^q + \dots + x_k^q) \leq (x_1 + \dots + x_k)^q \leq k^q(x_1^q + \dots + x_k^q). \quad (14)$$

LEMMA 2.4. *Let ν be a compact supported Borel probability measure on \mathbb{R}^d . Then for any $q > 0$,*

$$-d \leq \underline{\tau}(\nu, q) \leq dq. \quad (15)$$

Proof. Take a large number $R > 1$ such that ν is supported on the ball $B = B(0, R)$. Then for any $n \in \mathbb{N}$, there are at most $(3 \cdot 2^n R)^d$ many distinct $Q \in \mathbf{D}_n$ such that $Q \cap B \neq \emptyset$. Note that for any $q > 0$,

$$\tau_n(\nu, q) = \sum_{Q \in \mathbf{D}_n} \nu(Q)^q = \sum_{Q \in \mathbf{D}_n: Q \cap B \neq \emptyset} \nu(Q)^q.$$

Thus by (14),

$$\tau_n(\nu, q) \leq (3 \cdot 2^n R)^d \left(\sum_{Q \in \mathbf{D}_n: Q \cap B \neq \emptyset} \nu(Q) \right)^q = (3 \cdot 2^n R)^d$$

and

$$\tau_n(\nu, q) \geq (3 \cdot 2^n R)^{-dq} \left(\sum_{Q \in \mathbf{D}_n: Q \cap B \neq \emptyset} \nu(Q) \right)^q = (3 \cdot 2^n R)^{-dq}.$$

Therefore the lower L^q spectrum $\underline{\tau}(\nu, q) := \liminf_{n \rightarrow \infty} \frac{\log \tau_n(\nu, q)}{-n \log 2}$ satisfies (15). \square

LEMMA 2.5. *Let ν be a compactly supported Borel probability measure on \mathbb{R}^d , $m \in \mathbb{N}$ and $q > 0$. Suppose that $\{E_i\}_i$ is a collection of Borel subsets of \mathbb{R}^d such that $\nu(\bigcup_i E_i) = 1$. Furthermore assume that there is $N \in \mathbb{N}$ such that each E_i intersects at most N many elements in \mathbf{D}_m and any $Q \in \mathbf{D}_m$ intersects at most N many sets in $\{E_i\}_i$. Then*

$$N^{-(q+1)} \tau_m(\nu, q) \leq \sum_i \nu(E_i)^q \leq N^{q+1} \tau_m(\nu, q).$$

Proof. In the following we only prove that $\sum_i \nu(E_i)^q \leq N^{q+1} \tau_m(\nu, q)$, the other inequality $\tau_m(\nu, q) \leq N^{q+1} \sum_i \nu(E_i)^q$ can be proved symmetrically. By Lemma 2.3, we have

$$\begin{aligned} \sum_i \nu(E_i)^q &\leq \sum_i \left(\sum_{Q \in \mathbf{D}_m: Q \cap E_i \neq \emptyset} \nu(Q) \right)^q \leq \sum_i N^q \sum_{Q \in \mathbf{D}_m: Q \cap E_i \neq \emptyset} \nu(Q)^q \\ &\leq N^q \sum_{Q \in \mathbf{D}_m} \#\{i : E_i \cap Q \neq \emptyset\} \nu(Q)^q \leq N^{q+1} \tau_m(\nu, q), \end{aligned}$$

as desired. \square

LEMMA 2.6. *Let ν be a compactly supported Borel probability measure on \mathbb{R}^d . Let $q > 0$, $t > 1$ and $n \in \mathbb{N}$. Suppose that $\{B(x_i, t^{-1}2^{-n})\}_i$ is a family of disjoint balls in \mathbb{R}^d . Then*

$$\sum_i \nu(B(x_i, 2^{-n}))^q \leq \frac{1}{\gamma_d} 3^{dq} (5t)^d \tau_n(\nu, q),$$

where γ_d denotes the volume of the unit ball in \mathbb{R}^d .

Proof. Denote $B_i = B(x_i, 2^{-n})$. It is clear that each B_i intersects at most 2^d many cubes in \mathbf{D}_n . Conversely, since the family $\{B(x_i, t^{-1}2^{-n})\}_i$ is disjoint, each cube in \mathbf{D}_n intersects at most $\frac{(5t)^d}{\gamma_d}$ many balls in $\{B_i\}_i$. To see it, suppose a cube Q in \mathbf{D}_n intersects B_{i_1}, \dots, B_{i_k} . Then $5Q$, the cube which has side $5 \cdot 2^{-n}$ and has the same center as Q , contains B_{i_1}, \dots, B_{i_k} . Hence $5Q$ contains the disjoint balls $B(x_{i_j}, t^{-1}2^{-n})$, $j = 1, \dots, k$. It follows $k \leq \frac{(5t)^d}{\gamma_d}$ by comparing the volumes of $5Q$ and these disjoint balls. Therefore

$$\begin{aligned} \sum_i \nu(B_i)^q &\leq \sum_i \left(\sum_{Q \in \mathbf{D}_n, Q \cap B_i \neq \emptyset} \nu(Q) \right)^q \leq \sum_i 3^{dq} \sum_{Q \in \mathbf{D}_n, Q \cap B_i \neq \emptyset} \nu(Q)^q \\ &\leq 3^{dq} \sum_{Q \in \mathbf{D}_n} \left(\sum_{i: B_i \cap Q \neq \emptyset} 1 \right) \nu(Q)^q \leq \frac{3^{dq}(5t)^d}{\gamma_d} \tau_n(\nu, q), \end{aligned}$$

where we used (14) in the second inequality. This finishes the proof. \square

Proof of Proposition 2.1. Let $q \in (a, b)$. We divide the proof into three smaller steps.

Step 1. $\underline{d}(\nu, x) \geq \tau'(q+)$ for ν_q almost all $x \in \mathbb{R}^d$. It is sufficient to prove that for each $\gamma > 0$, $\underline{d}(\nu, x) \geq \tau'(q+) - \gamma$ for ν_q almost all $x \in \mathbb{R}^d$. To see it, fix $\gamma > 0$ and define

$$F_n = \left\{ x \in \mathbb{R}^d : \nu(B(x, 2^{-n})) \geq 2^{-n(\tau'(q+) - \gamma)} \right\}, \quad n \in \mathbb{N}.$$

We need to estimate $\nu_q(F_n)$ for large n . Since ν is compactly supported, F_n is bounded for each n . Set $\mathcal{B} = \{B(x, t^{-1}2^{-n}) : x \in F_n\}$, where t is the constant in (11). By Besicovitch's covering lemma, there exist families $\mathcal{B}_1, \dots, \mathcal{B}_c \subset \mathcal{B}$ covering F_n , where c only depends on d , such that the balls in each \mathcal{B}_j are disjoint. Take an j such that $\nu_q(\cup \mathcal{B}_j) \geq \frac{1}{c} \nu_q(F_n)$. For convenience, write $\mathcal{B}_j = \{B(x_i, t^{-1}2^{-n})\}_i$. Assume n is large enough such that $t^{-1}2^{-n} < r_0$. Then for any $u > 0$,

$$\begin{aligned} \nu_q(F_n) &\leq c \nu_q(\cup \mathcal{B}_j) = c \sum_i \nu_q(B(x_i, t^{-1}2^{-n})) \\ &\leq ch(2^{-n}) 2^{n\tau(q)} \sum_i \nu(B(x_i, 2^{-n}))^q \\ &\leq ch(2^{-n}) 2^{n\tau(q)} \sum_i \nu(B(x_i, 2^{-n}))^{q+u} \nu(B(x_i, 2^{-n}))^{-u} \\ &\leq ch(2^{-n}) 2^{n(\tau(q) + u\tau'(q+) - u\gamma)} \sum_i \nu(B(x_i, 2^{-n}))^{q+u}. \end{aligned}$$

By Lemma 2.6,

$$\nu_q(F_n) \leq \tilde{c} h(2^{-n}) 2^{n(\tau(q) + u\tau'(q+) - u\gamma)} \tau_n(\nu, q+u), \quad (16)$$

where $\tilde{c} = \frac{c3^{dq}(5t)^d}{\gamma_d}$. Choose a small $u > 0$ such that $q+u \in (a, b)$ and

$$\frac{\tau(q+u) - \tau(q)}{u} \geq \tau'(q) - \frac{\gamma}{4}.$$

Then choose a large integer N such that for $n > N$,

$$\frac{\log \tau_n(\nu, q+u)}{-n \log 2} \geq \tau(q+u) - \frac{u\gamma}{4}, \quad \check{c}h(2^{-n}) \leq 2^{u\gamma n/4}.$$

By (16) and a direct check, we have $\nu_q(F_n) < 2^{-u\gamma n/4}$ for $n > N$. Since $\nu_q(F_n)$ converges to 0 exponentially, we have $\nu_q(\overline{\lim} F_n) = 0$, where $\overline{\lim} F_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} F_n$. However the set $\{x : \underline{d}(\nu, x) < \tau'(q+) - \gamma\}$ is contained in $\overline{\lim} F_n$. Hence

$$\nu_q\{x \in \mathbb{R}^d : \underline{d}(\nu, x) < \tau'(q+) - \gamma\} = 0.$$

That is, $\underline{d}(\nu, x) \geq \tau'(q+) - \gamma$ for ν_q almost all $x \in \mathbb{R}^d$.

Step 2. $\overline{d}(\nu, x) \leq \tau'(q-)$ for ν_q almost all $x \in \mathbb{R}^d$. The statement is proved in a way similar to that in Step 1. In fact, we only need to prove that for any $\gamma > 0$, $\overline{d}(\nu, x) \leq \tau'(q-) + \gamma$ for ν_q almost all $x \in \mathbb{R}^d$. Fix $\gamma > 0$ and define

$$G_n = \left\{ x \in \text{supp}(\nu) : \nu(B(x, 2^{-n})) \leq 2^{-n(\tau'(q-)+\gamma)} \right\}, \quad n \in \mathbb{N}.$$

Similarly, by Bescovitch's covering lemma, we can find a family of disjoint balls $\{B(y_i, t^{-1}2^{-n})\}_i$ with $y_i \in G_n$, such that $\nu_q(G_n) \leq c\nu_q(\bigcup_i B(y_i, t^{-1}2^{-n}))$, where c is a constant that only depends on d . Assume that n is large enough such that $t^{-1}2^{-n} < r_0$. Then for $0 < u < q$,

$$\begin{aligned} \nu_q(G_n) &\leq c \sum_i \nu_q(B(y_i, t^{-1}2^{-n})) \leq ch(2^{-n})2^{n\tau(q)} \sum_i \nu(B(y_i, 2^{-n}))^q \\ &\leq ch(2^{-n})2^{n\tau(q)} \sum_i \nu(B(y_i, 2^{-n}))^{q-u} \nu(B(y_i, 2^{-n}))^u \\ &\leq ch(2^{-n})2^{n(\tau(q)-u\tau'(q-)-u\gamma)} \sum_i \nu(B(y_i, 2^{-n}))^{q-u} \\ &\leq \check{c}h(2^{-n})2^{n(\tau(q)-u\tau'(q-)-u\gamma)} \tau_n(\nu, q-u). \end{aligned} \quad (17)$$

Similarly, select a sufficient small $u > 0$ such that $q-u \in (a, b)$ and

$$\frac{\tau(q) - \tau(q-u)}{u} \leq \tau'(q-) + \frac{\gamma}{4}.$$

Then choose a large integer N such that for $n > N$,

$$\frac{\log \tau_n(\nu, q-u)}{-n \log 2} \geq \tau(q-u) - \frac{u\gamma}{4}, \quad \check{c}h(2^{-n}) \leq 2^{u\gamma n/4}.$$

By (17) and a direct check, we have $\nu_q(G_n) < 2^{-u\gamma n/4}$ for $n > N$. Thus $\nu_q(\overline{\lim} G_n) = 0$. However the set $\{x : \overline{d}(\nu, x) > \tau'(q-) + \gamma\}$ is contained in $\overline{\lim} G_n$. Hence $\overline{d}(\nu, x) \leq \tau'(q-) + \gamma$ for ν_q almost all $x \in \mathbb{R}^d$.

Step 3. When τ is differentiable at some $q > 0$, $d(\nu, x) = \tau'(q)$ for ν_q almost all $x \in \mathbb{R}^d$. Furthermore $\dim_H E_\nu(\alpha) = \tau^*(\alpha) = \alpha q - \tau(q)$ for $\alpha = \tau'(q)$. The first part of the statement follows directly from Step 1-2. Therefore we have $\nu_q(E_\nu(\alpha)) = 1$. To see the second part, by (11) we have $\underline{d}(\nu_q, x) \geq q\underline{d}(\nu, x) - \tau(q)$ for each $x \in \mathbb{R}^d$. Thus $\underline{d}(\nu_q, x) \geq q\alpha - \tau(q)$ for ν_q almost all $x \in \mathbb{R}^d$. Since $\nu_q(E_\nu(\alpha)) = 1$, by the Billingsley Theorem (c.f., [36, Theorem 6.9]), we have

$$\dim_H E_\nu(\alpha) \geq q\alpha - \tau(q).$$

Anyway the upper bound $\dim_H E_\nu(\alpha) \leq q\alpha - \tau(q)$ is generic whenever $E_\nu(\alpha) \neq \emptyset$ (see, e.g., Theorem 4.1 of [32]). Thus $\dim_H E_\nu(\alpha) = q\alpha - \tau(q) = \tau^*(\alpha)$. This finishes the proof of the proposition. \square

3. Some inequalities about self-conformal measures

In this section we prove the following inequalities about the self-conformal measures, which will be used in our proof of Theorem 1.2 about the Gibbs properties of self-conformal measures.

PROPOSITION 3.1. *Let ν be a self-conformal measure on \mathbb{R}^d . Then*

- (i) *For any $q \geq 1$, there exists a sequence (c_n) of positive numbers such that $\lim_{n \rightarrow \infty} \frac{\log c_n}{n} = 0$ and, for any $m, n \in \mathbb{N}$ and $\tilde{Q} \in \mathbf{D}_n$,*

$$\sum_{Q \in \mathbf{D}_{m+n}: Q \subset \tilde{Q}} \nu(Q)^q \leq c_n \tau_m(\nu, q) \sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \nu(\tilde{B})^q, \quad (18)$$

where $\tilde{B} \sim \tilde{Q}$ means that the closures of \tilde{B} and \tilde{Q} intersect.

- (ii) *For any $0 < q < 1$, there exists a sequence (c_n) of positive numbers such that $\lim_{n \rightarrow \infty} \frac{\log c_n}{n} = 0$ and, for any $m, n \in \mathbb{N}$ and $\tilde{Q} \in \mathbf{D}_n$,*

$$\sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \left(\sum_{Q \in \mathbf{D}_{m+n}: Q \subset \tilde{B}} \nu(Q)^q \right) \geq c_n \tau_m(\nu, q) \nu(\tilde{Q})^q. \quad (19)$$

We point out that the above proposition was first proved by Peres and Solomyak [44, pp 1609-1612] under an additional assumption that the generating IFS for ν satisfies the bounded distortion property (4). Indeed, under that assumption, Peres and Solomyak obtained a slight stronger result that the sequence (c_n) in Proposition 3.1 can be replaced by a uniform constant, and they used these inequalities to show the existence of the limit $\tau(\nu, q)$ in (2).

Let $\{S_i\}_{i=1}^\ell$ be a C^1 -conformal IFS on a compact set $X \subset \mathbb{R}^d$. Assume that each S_i extends to an injective contracting C^1 -conformal map $S_i : U \rightarrow U$ on an open set $U \supset X$. Denote by K the corresponding self-conformal set. Let ν be the self-conformal measure satisfying (3). Let $A = \{1, \dots, \ell\}$. Denote $A^* = \bigcup_{n \geq 1} A^n$. For $u = u_1 \dots u_k$, we write $S_u = S_{u_1} \circ \dots \circ S_{u_k}$, $p_u = p_{u_1} \dots p_{u_k}$ and $K_u = S_u(K)$; in particular we let \tilde{u} denote the word obtained by dropping the last letter of u . For $n \in \mathbb{N}$, denote

$$W_n := \{u \in A^* : \text{diam}(K_u) \leq 2^{-n}, \text{diam}(K_{\tilde{u}}) > 2^{-n}\}. \quad (20)$$

To prove Proposition 3.1, we need the following lemma proved by Peres and Solomyak.

LEMMA 3.2. (Peres and Solomyak). *Let ν be a self-conformal measure on \mathbb{R}^d . Then*

(i) For any $q \geq 1$, there exists $C > 0$ such that for any $m, n \in \mathbb{N}$ and $\tilde{Q} \in \mathbf{D}_n$,

$$\sum_{Q \in \mathbf{D}_{m+n}: Q \subset \tilde{Q}} \nu(Q)^q \leq C \max_{u \in W_n} \sum_{Q \in \mathbf{D}_{m+n}} \nu(S_u^{-1}(Q))^q \sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \nu(\tilde{B})^q, \quad (21)$$

(ii) For any $0 < q < 1$, there exists $C > 0$ such that for any $m, n \in \mathbb{N}$ and $\tilde{Q} \in \mathbf{D}_n$,

$$\sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \left(\sum_{Q \in \mathbf{D}_{m+n}: Q \subset \tilde{B}} \nu(Q)^q \right) \geq C \nu(\tilde{Q})^q \min_{u \in W_n} \sum_{Q \in \mathbf{D}_{m+n}} \nu(S_u^{-1}(Q))^q. \quad (22)$$

We remark that the above lemma was proved implicitly in the proof of [44, Theorem 1.1], depending only on the convexity of x^q (when $q \geq 1$); the concavity of x^q (when $0 < q < 1$) and the self-similar relation:

$$\nu = \sum_{u \in W_n} p_u \nu \circ S_u^{-1}, \quad \text{for all } n \in \mathbb{N}. \quad (23)$$

One can check that Proposition 3.1 follows directly from Lemma 3.2 and the following proposition:

PROPOSITION 3.3. *There exists $\beta > 0$ such that for any $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that for all $q > 0$, $m, n \in \mathbb{N}$, and all $u \in W_n$,*

$$(C(\epsilon)(1 + \epsilon)^{\beta n})^{-(q+1)} \tau_m(\nu, q) \leq \sum_{Q \in \mathbf{D}_{m+n}} \nu(S_u^{-1}Q)^q \leq (C(\epsilon)(1 + \epsilon)^{\beta n})^{q+1} \tau_m(\nu, q).$$

In the remaining part of this section, we will provide a detailed proof of the above proposition. We remark that in Proposition 3.3 the measure ν can be replaced by any probability measures supported on K . Furthermore if the bounded distortion assumption is fulfilled, then there is a constant $C > 0$ (depending on q) such that $C^{-1} \tau_m(\nu, q) \leq \nu(S_u^{-1}Q)^q \leq C \tau_m(\nu, q)$, as proved by Peres and Solomyak.

To prove Proposition 3.3, we first prove the following elementary result.

LEMMA 3.4. *For any $c > 1$, there exists $\delta > 0$ such that*

$$c^{-1} |S'_i(x)| \cdot |x - y| \leq |S_i(x) - S_i(y)| \leq c |S'_i(x)| \cdot |x - y| \quad (24)$$

for all $1 \leq i \leq \ell$, $x \in X$, $y \in U$ with $|x - y| \leq \delta$.

Proof. Without loss of generality we show (24) for the case $i = 1$. Denote $S = S_1$. Assume the result is false. Then there exists $c > 1$, and two sequences $(x_n) \subset X$, $(y_n) \subset U$ such that $x_n \neq y_n$, $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$ and

$$|S(x_n) - S(y_n)| \geq c |S'(x_n)| \cdot |x_n - y_n| \quad \text{or} \quad |S(x_n) - S(y_n)| \leq c^{-1} |S'(x_n)| \cdot |x_n - y_n|. \quad (25)$$

Since X is compact, without lost of generality, we assume that $\lim_{n \rightarrow \infty} x_n = x = \lim_{n \rightarrow \infty} y_n$. Write $S = (f_1, f_2, \dots, f_d)^t$. Then each component f_j of S is a C^1 real valued function defined on U . Choose a small $\epsilon > 0$ such that

$$B_\epsilon(X) := \{z \in \mathbb{R}^d : |z - x| \leq \epsilon \text{ for some } x \in X\} \subset U.$$

Take $N \in \mathbb{N}$ such that $|x_n - y_n| < \epsilon$ for $n \geq N$. By the mean value theorem, for each $n \geq N$ and $1 \leq j \leq d$, there exists $z_{n,j}$ on the segment L_{x_n, y_n} connecting x_n and y_n such that

$$f_j(x_n) - f_j(y_n) = \nabla f_j(z_{n,j}) \cdot (x_n - y_n),$$

where ∇f_j denote the gradient of f_j . Therefore $|S(x_n) - S(y_n)| = |M_n(x_n - y_n)|$ with $M_n := (\nabla f_1(z_{n,1}), \dots, \nabla f_d(z_{n,d}))^t$. Since S is C^1 , M_n turns to $S'(x)$ as $n \rightarrow \infty$. Note that $S'(x)$ is the product of a positive scalar and an orthogonal matrix, we have $\lim_{n \rightarrow \infty} \frac{|M_n(x_n - y_n)|}{|x_n - y_n|} = |S'(x)|$, leading to a contradiction with (25). \square

Take $\epsilon_0 > 0$ such that $B_{\epsilon_0}(X) \subset U$. Then we have the following result concerning with the distortion property of the IFS $\{S_i\}_{i=1}^\ell$.

LEMMA 3.5. *For any $c > 1$, there exists $D_1 > 0$ such that for any $u \in \{1, \dots, \ell\}^n$, $x \in X$ and $y \in B_{\epsilon_0}(X)$, we have*

$$D_1^{-1} c^{-n} |S'_u(x)| \cdot |x - y| \leq |S_u(x) - S_u(y)| \leq D_1 c^n |S'_u(x)| \cdot |x - y|. \quad (26)$$

Proof. Since $S_i : U \rightarrow U$ ($i = 1, \dots, \ell$) are contractive, there exists $0 < \gamma < 1$ such that for all $1 \leq i \leq \ell$,

$$|S_i(z_1) - S_i(z_2)| \leq \gamma |z_1 - z_2|, \quad \forall z_1, z_2 \in U. \quad (27)$$

Denote $a = \inf\{|S'_i(z)| : z \in X, 1 \leq i \leq \ell\}$ and $b = \sup\{|S'_i(z)| : z \in X, 1 \leq i \leq \ell\}$. Then $0 < a \leq b \leq \gamma < 1$.

To prove the lemma, we may assume without loss of generality that $1 < c < 1/b$. By Lemma 3.4, there exists $\delta > 0$ (depending on c), such that for any $i = 1, \dots, \ell$, $z_1 \in X$, $z_2 \in U$ with $|z_1 - z_2| \leq \delta$, we have

$$c^{-1} |S'_i(z_1)| \cdot |z_1 - z_2| \leq |S_i(z_1) - S_i(z_2)| \leq c |S'_i(z_1)| \cdot |z_1 - z_2| \leq \delta. \quad (28)$$

Choose a large positive integer N such that $\gamma^N \cdot \text{diam}(B_{\epsilon_0}(X)) < \delta$. Denote

$$\eta_1 = \inf \left\{ \frac{|S_v(z_1) - S_v(z_2)|}{|S'_v(z_1)| \cdot |z_1 - z_2|} : |v| \leq N, z_1 \in X, z_2 \in B_{\epsilon_0}(X) \text{ with } |z_1 - z_2| \geq \delta \right\}$$

and

$$\eta_2 = \sup \left\{ \frac{|S_v(z_1) - S_v(z_2)|}{|S'_v(z_1)| \cdot |z_1 - z_2|} : |v| \leq N, z_1 \in X, z_2 \in B_{\epsilon_0}(X) \text{ with } |z_1 - z_2| \geq \delta \right\}.$$

A compactness argument, together with the fact that S_i 's are injective on U , shows that $0 < \eta_1 \leq \eta_2 < \infty$.

Now let $u = u_1 \dots u_n \in \{1, \dots, \ell\}^n$, $x \in X$ and $y \in B_{\epsilon_0}(X)$. We estimate $|S_u(x) - S_u(y)|$ in the following three different possible cases respectively: (i) $|x - y| < \delta$; (ii) $|x - y| \geq \delta$ and $n \leq N$; (iii) $|x - y| \geq \delta$ and $n > N$.

If (i) occurs, then using (28) repeatedly, we have

$$c^{-n} |S'_u(x)| \cdot |x - y| \leq |S_u(x) - S_u(y)| \leq c^n |S'_u(x)| \cdot |x - y|. \quad (29)$$

Meanwhile if (ii) occurs, then by the definition of η_1, η_2 we have

$$\eta_1 |S'_u(x)| \cdot |x - y| \leq |S_u(x) - S_u(y)| \leq \eta_2 |S'_u(x)| \cdot |x - y|. \quad (30)$$

In the end we assume that (iii) occurs. Since $|x - y| \leq \text{diam}(B_{\epsilon_0}(X))$, we have

$$|S_{u_{n-N+1}\dots u_n}(x) - S_{u_{n-N+1}\dots u_n}(y)| \leq \gamma^N \text{diam}(B_{\epsilon_0}(X)) \leq \delta.$$

Therefore by using (28) repeatedly, we have

$$c^{-(n-N)} t |S'_{u_1\dots u_{n-N}}(z)| \leq |S_u(x) - S_u(y)| \leq c^{n-N} t |S'_{u_1\dots u_{n-N}}(z)|, \quad (31)$$

where $z := S_{u_{n-N+1}\dots u_n}(x)$ and $t := |S_{u_{n-N+1}\dots u_n}(x) - S_{u_{n-N+1}\dots u_n}(y)|$. By the definition of η_1, η_2 , we have $a^N \eta_1 |x - y| \leq t \leq b^N \eta_2 |x - y|$. Furthermore

$$|S'_u(x)| / |S'_{u_1\dots u_{n-N}}(z)| = |S'_{u_{n-N+1}\dots u_n}(x)| \in [a^N, b^N].$$

Applying these estimates in (31) yields

$$c^{-(n-N)} (a/b)^N \eta_1 |S'_u(x)| \cdot |x - y| \leq |S_u(x) - S_u(y)| \leq c^{n-N} (b/a)^N \eta_2 |S'_u(x)| \cdot |x - y|. \quad (32)$$

Combining the estimates (29), (30) and (32) yields the desired result. \square

As a corollary of the above result, we have

COROLLARY 3.6. *For any $c > 1$, there exists $D_2 > 0$ such that for any $u \in \{1, \dots, \ell\}^n$ and $x \in K$,*

$$D_2^{-1} c^{-n} |S'_u(x)| \leq \text{diam}(S_u(K)) \leq D_2 c^n |S'_u(x)|.$$

Proof. Let $c > 1$ be given and let $D_1 > 0$ be the constant in Lemma 3.5. Fix $x \in K \subset X$. Take $y \in K$ such that $\text{diam}(S_u(K)) \leq 2|S_u(x) - S_u(y)|$. By Lemma 3.5,

$$|S_u(x) - S_u(y)| \leq D_1 c^n |S'_u(x)| \cdot |x - y| \leq D_1 c^n |S'_u(x)| \text{diam}(K).$$

Hence we have $\text{diam}(S_u(K)) \leq 2D_1 c^n |S'_u(x)| \text{diam}(K)$. To see the other inequality, choose $z \in K$ such that $\text{diam}(K) \leq 2|x - z|$. Again by Lemma 3.5, we have

$$\begin{aligned} \text{diam}(S_u(K)) &\geq |S_u(x) - S_u(z)| \geq D_1^{-1} c^{-n} |S'_u(x)| \cdot |x - z| \\ &\geq 2^{-1} D_1^{-1} c^{-n} |S'_u(x)| \text{diam}(K). \end{aligned}$$

This finishes the proof of the corollary. \square

LEMMA 3.7. *There exist $\alpha_1, \alpha_2 > 0$ such that for any $n \in \mathbb{N}$ and $u \in W_n$, we have (i) $|u| \leq \alpha_1 n$; (ii) $\text{diam}(K_u) \geq \alpha_2 2^{-n}$.*

Proof. Let γ be defined as in (27). Then $2^{-n} \leq \text{diam}(K_{\bar{u}}) \leq \gamma^{|u|} \text{diam}(K)$. Hence $|u| \leq 1 + \frac{n \log 2}{\log(1/\gamma)} + \frac{\log \text{diam}(K)}{\log \gamma}$. Thus (i) follows by setting $\alpha_1 = 1 + \frac{\log 2}{\log(1/\gamma)} + \frac{|\log \text{diam}(K)|}{|\log \gamma|}$. To see (ii), set

$$\alpha_2 = \inf \left\{ \frac{|S_i(x) - S_i(y)|}{|x - y|} : x, y \in K, x \neq y, 1 \leq i \leq \ell \right\}.$$

A compactness argument shows that $\alpha_2 > 0$. Hence $\text{diam}(K_u) \geq \alpha_2 \text{diam}(K_{\bar{u}}) \geq \alpha_2 2^{-n}$. \square

PROPOSITION 3.8. *There exists a constant $\beta_1 > 0$ such that for any $c > 1$, there exists $D_3 > 0$ (depending on c) such that for any $n, m \in \mathbb{N}$, $Q \in \mathbf{D}_{m+n}$ and $u \in W_n$, if $S_u^{-1}(Q) \cap K \neq \emptyset$, then*

- (i) *$\text{diam}(S_u^{-1}(Q) \cap B_{\epsilon_0}(X)) \leq D_3 c^{\beta_1 n} 2^{-m}$.*
- (ii) *The set $S_u^{-1}(Q^*) \cap B_{\epsilon_0}(X)$ contains a ball of radius $D_3^{-1} c^{-\beta_1 n} 2^{-m}$, where Q^* denotes the cube of side 2^{-m-n+1} with the same center as Q .*

Proof. Let $x \in S_u^{-1}(Q) \cap K$ and $y \in B_{\epsilon_0}(X)$. Then by Lemma 3.5,

$$D_1^{-1} c^{-|u|} |S'_u(x)| \cdot |x - y| \leq |S_u(x) - S_u(y)| \leq \text{diam}(Q) \leq \sqrt{d} 2^{-m-n}.$$

By Corollary 3.6, we have $|S'_u(x)| \geq D_2^{-1} c^{-|u|} \text{diam}(K_u)$. Combining these inequalities we have

$$\begin{aligned} |x - y| &\leq D_1 D_2 c^{2|u|} \sqrt{d} 2^{-m-n} (\text{diam}(K_u))^{-1} \\ &\leq (\alpha_1)^{-1} \sqrt{d} D_1 D_2 c^{2\alpha_1 n} 2^{-m}, \quad (\text{by Lemma 3.7}) \end{aligned}$$

from which (i) follows.

To see (ii), let $r := \epsilon_0 (D_1 D_2)^{-1} c^{-2\alpha_1 n} 2^{-m}$. Let $z \in B_r(x) \subset B_{\epsilon_0}(X)$. Then by Lemma 3.5, Corollary 3.6 and Lemma 3.7, we have

$$\begin{aligned} |S_u(x) - S_u(z)| &\leq D_1 c^{|u|} |S'_u(x)| \cdot |x - z| \leq D_1 D_2 c^{2|u|} \text{diam}(K_u) r \\ &\leq D_1 D_2 c^{2\alpha_1 n} 2^{-n} r \leq 2^{-m-n}. \end{aligned}$$

Since $S_u(x) \in Q$, we have $S_u(z) \in Q^*$, i.e., $z \in S_u^{-1}(Q^*)$. Since z is arbitrarily taken from $B_r(x)$, we deduce that $S_u^{-1}(Q^*) \cap B_{\epsilon_0}(X)$ contains the ball $B_r(x)$, as desired. \square

Applying the above proposition together with a simple geometric argument we obtain directly

COROLLARY 3.9. *There exists a constant $\beta_2 > 0$ such that for any $c > 1$, there exists $D_4 > 0$ (depending on c) such that for any $n, m \in \mathbb{N}$, and $u \in W_n$, if we denote*

$$\mathcal{F} := \{S_u^{-1}(Q) \cap B_{\epsilon_0}(X) : Q \in \mathbf{D}_{m+n}, S_u^{-1}(Q) \cap K \neq \emptyset\},$$

then (i) each $E \in \mathcal{F}$ intersects at most $D_4 c^{\beta_2 n}$ cubes in \mathbf{D}_m ; (ii) each cubes in \mathbf{D}_m intersects at most $D_4 c^{\beta_2 n}$ sets in \mathcal{F} .

Proof of Proposition 3.3. By Corollary 3.9 and Lemma 2.5, for any $q > 0$ we have

$$(D_4 c^{\beta_2 n})^{-(q+1)} \tau_m(\nu, q) \leq \sum_{Q \in \mathbf{D}_{m+n}} \nu(S_u^{-1}Q)^q \leq (D_4 c^{\beta_2 n})^{q+1} \tau_m(\nu, q),$$

where we have used

$$\sum_{Q \in \mathbf{D}_{m+n}} \nu(S_u^{-1}Q)^q = \sum_{Q \in \mathbf{D}_{m+n} : S_u^{-1}Q \cap K \neq \emptyset} \nu(S_u^{-1}Q \cap B_{\epsilon_0}(X))^q.$$

This proves the proposition. \square

REMARK 3.10. Proposition 3.1 hold for a broader class of probability measures on self-conformal sets rather than the strict self-conformal measures. Indeed, let μ be a Borel probability measure on the one-sided full shift space $\Sigma = \{1, \dots, \ell\}^{\mathbb{N}}$. Assume that μ satisfies the inequality (9), and let $\nu = \mu \circ \pi^{-1}$ be the projection of μ on \mathbb{R}^d by a canonical projection π generated by a conformal IFS $\{S_i\}_{i=1}^{\ell}$ on \mathbb{R}^d (see (8) for the definition). Then it is not hard to show that for any Borel set $E \subset \mathbb{R}^d$,

$$\nu(E) \leq C \sum_{u \in G} \mu([u]) \nu(S_u^{-1}(E)), \quad (33)$$

where G is an arbitrary finite set of words over $\{1, \dots, \ell\}$ such that $\Sigma = \bigcup_{u \in G} [u]$. Then one can use the above inequality, instead of the strict self-similar relation (3), to modify the proof of Peres and Solomyak slightly to obtain (21). It together with Proposition 3.3, which is valid for all probability measures supported on K rather than ν , yields (18). Similarly if μ satisfies the inequality (10), then for any Borel set $E \subset \mathbb{R}^d$,

$$\nu(E) \geq C' \sum_{u \in G} \mu([u]) \nu(S_u^{-1}(E)), \quad (34)$$

where G is an arbitrary finite set of words over $\{1, \dots, \ell\}$ such that $\{\Sigma, u \in G\}$ is a partition of Σ . Then one can use it to modify the proof of Peres and Solomyak to obtain (22). It together with Proposition 3.3 yields (19).

4. Gibbs properties of self-conformal measures

In this section we set up the Gibbs properties of self-conformal measures. We first present some elementary results.

LEMMA 4.1. *Assume that $(a_n), (c_n)$ are two sequences of positive numbers satisfying $\lim_{n \rightarrow \infty} \frac{1}{n} \log c_n = 0$. Then we have the following statements.*

- (i) *If $a_{n+m} \leq c_n a_n a_m$ for all $m, n \in \mathbb{N}$, then the limit $a = \lim_{n \rightarrow \infty} \frac{\log a_n}{n}$ exists with $a \in [-\infty, \infty)$.*
- (ii) *If $a_{n+m} \geq c_n a_n a_m$ for all $m, n \in \mathbb{N}$, then the limit $a = \lim_{n \rightarrow \infty} \frac{\log a_n}{n}$ exists with $a \in (-\infty, \infty]$.*

Proof. We only prove (i), whilst (ii) can be proved in a similar way. Fix $\ell \in \mathbb{N}$. Then for any $k \in \mathbb{N}$ and $0 \leq r < \ell$,

$$a_{k\ell+r} \leq (c_\ell a_\ell)^k a_r.$$

Hence

$$\frac{\log a_{k\ell+r}}{k\ell+r} \leq \frac{k(\log c_\ell + \log a_\ell)}{k\ell+r} + \frac{\log a_r}{k\ell+r}.$$

Letting $k \uparrow \infty$, we have

$$\limsup_{n \rightarrow \infty} \frac{\log a_n}{n} \leq \frac{\log c_\ell + \log a_\ell}{\ell} < \infty.$$

Taking $\ell \uparrow \infty$, we have $\limsup_{n \rightarrow \infty} \frac{\log a_n}{n} \leq \liminf_{\ell \rightarrow \infty} \frac{\log a_\ell}{\ell}$. This finishes the proof. \square

LEMMA 4.2. *Assume that (a_n) is a sequence of positive numbers with $\lim_{n \rightarrow \infty} \frac{1}{n} \log a_n = a \in \mathbb{R}$. Then for any $\epsilon > 0$, there exists a monotone decreasing sequence (t_n) of positive numbers such that $t_1 = \epsilon$, $\lim_{n \rightarrow \infty} t_n = 0$ and*

$$\sum_{n=1}^{\infty} a_n b_n e^{-na} = \infty,$$

where $b_n = e^{t_1 + \dots + t_n}$.

Proof. We construct (t_n) by induction. Define $t_1 = \epsilon$. Assume that t_1, \dots, t_n have been defined well. Set $b_n = e^{t_1 + \dots + t_n}$. Then define

$$t_{n+1} = \begin{cases} \frac{t_n}{2}, & \text{if } a_n b_n e^{-na} \geq 1, \\ t_n, & \text{otherwise.} \end{cases}$$

It is readily easy to see that the sequence (t_n) satisfies the desired properties. \square

The key results in this section are Proposition 4.3 and Proposition 4.4. Our proofs are inspired by Michon and Peyrière's construction of Gibbs measures for homogeneous trees (see [38, 48]) and Testud's extension for measures satisfying (9) (see [56]).

PROPOSITION 4.3. *Let ν be a compactly supported Borel probability measure on \mathbb{R}^d . Given $q > 0$, assume that there is a sequence (c_n) of positive numbers such that $\lim_{n \rightarrow \infty} \frac{\log c_n}{n} = 0$ and, for any $n, m \in \mathbb{N}$, any $\tilde{Q} \in \mathbf{D}_n$,*

$$\sum_{Q \in \mathbf{D}_{n+m}: Q \subset \tilde{Q}} \nu(Q)^q \leq c_n \tau_m(\nu, q) \sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \nu(\tilde{B})^q, \quad (35)$$

where $\tilde{B} \sim \tilde{Q}$ means that the closures of \tilde{B} and \tilde{Q} intersect. Then the limit $\tau(q) = \tau(\nu, q)$ in (2) exists. Furthermore, there exists a Borel probability measure ν_q on \mathbb{R}^d such that for any $n \in \mathbb{N}$ and $\tilde{Q} \in \mathbf{D}_n$,

$$\nu_q(\text{int}(\tilde{Q})) \leq c_n 2^{n\tau(q)} \sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \nu(\tilde{B})^q, \quad (36)$$

where $\text{int}(\tilde{Q})$ denotes the interior of \tilde{Q} . Moreover, for any $x \in \mathbb{R}^d$ and $0 < r < 1$,

$$\nu_q(B(x, t^{-1}r)) \leq h(r)r^{-\tau(q)} \nu(B(x, r))^q, \quad (37)$$

where $t = \frac{1}{16\sqrt{d}}$ and $h(r) = 6^d 4^{|\tau(q)|} \max\{c_n : n \leq \log_2(8\sqrt{d}/r)\}$.

Proof. Observe that for any $\tilde{B} \in \mathbf{D}_n$, $\tilde{B} \sim \tilde{Q}$ for at most 3^d many distinct $\tilde{Q} \in \mathbf{D}_n$. Summing over $\tilde{Q} \in \mathbf{D}_n$ in (35) yields

$$\tau_{m+n}(\nu, q) \leq 3^q c_n \tau_n(\nu, q) \tau_m(\nu, q).$$

By Lemma 4.1, $\tau(q) = \tau(\nu, q)$ exists with $\tau(q) \in (-\infty, \infty]$. By Lemma 2.4, $\tau(q) < \infty$.

Take a large positive number R such that ν is supported in the closed ball $B(0, R)$. In the following, for each $\epsilon > 0$ we construct a Borel probability measure μ_ϵ supported on $B(0, R + \sqrt{d})$. By Lemma 4.2, we can construct a decreasing sequence $(t_{\epsilon, n})_{n=1}^\infty$ of positive numbers such that $t_{\epsilon, 1} = \epsilon$, $\lim_{n \rightarrow \infty} t_{\epsilon, n} = 0$, and

$$\sum_{n=1}^{\infty} b_{\epsilon, n} 2^{n\tau(q)} \tau_n(\nu, q) = \infty, \quad (38)$$

where $b_{\epsilon, n} = e^{t_{\epsilon, 1} + \dots + t_{\epsilon, n}}$. It is clear that $b_{\epsilon, n} \leq e^{\epsilon n}$ and $b_{\epsilon, n+m} \leq b_{\epsilon, n} b_{\epsilon, m}$ for all m, n . For $s \in (-\infty, \tau(q)]$, define

$$Z_\epsilon(s) = \sum_{n=1}^{\infty} b_{\epsilon, n} 2^{ns} \tau_n(\nu, q).$$

By (38), $Z_\epsilon(s) < \infty$ for $s < \tau(q)$, and $Z_\epsilon(\tau(q)) = \lim_{s \uparrow \tau(q)} Z_\epsilon(s) = \infty$.

Let $s < \tau(q)$. Define $\phi_{\epsilon, s} : \mathbb{R}^d \rightarrow [0, \infty]$ by

$$\phi_{\epsilon, s}(x) = \sum_{n=1}^{\infty} b_{\epsilon, n} 2^{n(s+d)} \nu(\tilde{Q}_n(x))^q,$$

where $\tilde{Q}_n(x)$ denotes the cube in \mathbf{D}_n which contains x . It is clear that the function $\phi_{\epsilon, s}$ is Borel measurable. A direct calculation shows that

$$\int_{\mathbb{R}^d} \phi_{\epsilon, s}(x) dx = Z_\epsilon(s).$$

Thus $\phi_{\epsilon, s} \in L^1(\mathbb{R}^d)$. Define a measure $\mu_{\epsilon, s}$ on \mathbb{R}^d by

$$\mu_{\epsilon, s}(E) = \frac{1}{Z_\epsilon(s)} \int_E \phi_{\epsilon, s}(x) dx \quad \text{for Borel } E \subset \mathbb{R}^d.$$

Then $\mu_{\epsilon, s}$ is a Borel probability measure supported on $B(0, R + \sqrt{d})$.

For any $n \in \mathbb{N}$ and any $\tilde{Q} \in \mathbf{D}_n$, let \tilde{Q}_k ($1 \leq k \leq n$) denotes the cube in \mathbf{D}_k that contains \tilde{Q} . Then a direct check shows

$$\mu_{\epsilon, s}(\tilde{Q}) = Z_\epsilon(s)^{-1} \int_{\tilde{Q}} \phi_{\epsilon, s}(x) dx = (I) + (II),$$

where

$$(I) := Z_\epsilon(s)^{-1} 2^{-nd} \left(\sum_{k=1}^n b_{\epsilon, k} 2^{k(s+d)} \nu(\tilde{Q}_k)^q \right)$$

and

$$(II) := Z_\epsilon(s)^{-1} \sum_{m=1}^{\infty} b_{\epsilon, n+m} 2^{(n+m)s} \sum_{Q \in \mathbf{D}_{n+m}: Q \subset \tilde{Q}} \nu(Q)^q.$$

Using $b_{\epsilon, n+m} \leq b_{\epsilon, n} b_{\epsilon, m}$ and (35), we have

$$(II) \leq c_n b_{\epsilon, n} 2^{ns} \sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \nu(\tilde{B})^q. \quad (39)$$

Therefore

$$\mu_{\epsilon,s}(\tilde{Q}) \leq Z_\epsilon(s)^{-1} 2^{-nd} \left(\sum_{k=1}^n b_{\epsilon,k} 2^{k(s+d)} \nu(\tilde{Q}_k)^q \right) + c_n b_{\epsilon,n} 2^{ns} \sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \nu(\tilde{B})^q. \quad (40)$$

Let μ_ϵ be a limit point of $(\mu_{\epsilon,s})$ in the weak-star topology as $s \uparrow \tau(q)$. By (40) and using the fact $\lim_{s \uparrow \tau(q)} Z_\epsilon(s) = \infty$, we have

$$\mu_\epsilon(\text{int}(\tilde{Q})) \leq \limsup_{s \uparrow \tau(q)} \mu_{\epsilon,s}(\tilde{Q}) \leq c_n b_{\epsilon,n} 2^{n\tau(q)} \sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \nu(\tilde{B})^q.$$

Let μ be a limit point of (μ_ϵ) in the weak-star topology as $\epsilon \rightarrow 0$. Since $b_{\epsilon,n} \leq 2^{n\epsilon}$, we have

$$\mu(\text{int}(\tilde{Q})) \leq \limsup_{\epsilon \rightarrow 0} \mu_\epsilon(\text{int}(\tilde{Q})) \leq c_n 2^{n\tau(q)} \sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \nu(\tilde{B})^q, \quad \forall \tilde{Q} \in \mathbf{D}_n.$$

This finishes the proof of (36) by letting $\nu_q = \mu$.

To prove (37), let the measures $\mu_{\epsilon,s}$, μ_ϵ and μ be constructed as above. Let $x \in \mathbb{R}^d$ and $0 < r < 1$. Set $\tilde{r} = \frac{r}{12\sqrt{d}}$. Choose $n \in \mathbb{N}$ such that $2\tilde{r} < 2^{-n} \leq 4\tilde{r}$. Then

$$B\left(x, \frac{r}{16\sqrt{d}}\right) \subset \text{int}(B(x, \tilde{r})) \subset \text{int}\left(\bigcup_{\tilde{Q} \in \mathbf{D}_n, \tilde{Q} \sim B(x, \tilde{r})} \tilde{Q}\right). \quad (41)$$

By (40), for each $\tilde{Q} \in \mathbf{D}_n$ with $\tilde{Q} \sim B(x, \tilde{r})$, we have

$$\limsup_{s \uparrow \tau(q)} \mu_{\epsilon,s}(\tilde{Q}) \leq c_n b_{\epsilon,n} 2^{n\tau(q)} \sum_{\tilde{B} \sim \tilde{Q}} \nu(\tilde{B})^q.$$

It is easy to see that for all those $\tilde{B} \in \mathbf{D}_n$ with $\tilde{B} \sim \tilde{Q}$, we have $\tilde{B} \subset B(x, 3\sqrt{d} \cdot 2^{-n}) \subset B(x, r)$. Since there are at most 3^d different many \tilde{B} 's in \mathbf{D}_n such that $\tilde{B} \sim \tilde{Q}$, we have

$$\limsup_{s \uparrow \tau(q)} \mu_{\epsilon,s}(\tilde{Q}) \leq c_n b_{\epsilon,n} 2^{n\tau(q)} 3^d \nu(B(x, r))^q.$$

Summing over all $\tilde{Q} \in \mathbf{D}_n$ with $\tilde{Q} \sim B(x, \tilde{r})$ (noting that there are at most 2^d such \tilde{Q} 's), we have

$$\limsup_{s \uparrow \tau(q)} \mu_{\epsilon,s} \left(\bigcup_{\tilde{Q} \in \mathbf{D}_n, \tilde{Q} \sim B(x, \tilde{r})} \tilde{Q} \right) \leq c_n b_{\epsilon,n} 2^{n\tau(q)} 6^d \nu(B(x, r))^q.$$

Therefore

$$\mu_\epsilon \left(\text{int} \left(\bigcup_{\tilde{Q} \in \mathbf{D}_n, \tilde{Q} \sim B(x, \tilde{r})} \tilde{Q} \right) \right) \leq c_n b_{\epsilon,n} 2^{n\tau(q)} 6^d \nu(B(x, r))^q.$$

Hence by (41), we have

$$\begin{aligned} \mu\left(B\left(x, \frac{r}{16\sqrt{d}}\right)\right) &\leq \mu(\text{int}(B(x, \tilde{r}))) \leq \mu\left(\text{int}\left(\bigcup_{\tilde{Q} \in \mathbf{D}_n, \tilde{Q} \sim B(x, \tilde{r})} \tilde{Q}\right)\right) \\ &\leq \limsup_{\epsilon \rightarrow 0} \mu_\epsilon\left(\text{int}\left(\bigcup_{\tilde{Q} \in \mathbf{D}_n, \tilde{Q} \sim B(x, \tilde{r})} \tilde{Q}\right)\right) \\ &\leq c_n 2^{n\tau(q)} 6^d \nu(B(x, r))^q, \end{aligned}$$

From which (37) follows. \square

As an analogue of Proposition 4.3, we have

PROPOSITION 4.4. *Let ν be a compactly supported Borel probability measure on \mathbb{R}^d . Given $q > 0$, assume that there is a sequence (c_n) of positive numbers such that $\lim_{n \rightarrow \infty} \frac{\log c_n}{n} = 0$ and, for any $n, m \in \mathbb{N}$, any $\tilde{Q} \in \mathbf{D}_n$,*

$$\sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \left(\sum_{Q \in \mathbf{D}_{n+m}: Q \subset \tilde{B}} \nu(Q)^q \right) \geq c_n \tau_m(\nu, q) \nu(\tilde{Q})^q. \quad (42)$$

Then $\tau(q) = \tau(\nu, q)$ exists. Furthermore, there exists a Borel probability measure ν_q on \mathbb{R}^d such that for any $n \in \mathbb{N}$ and $\tilde{Q} \in \mathbf{D}_n$,

$$\sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \nu_q(\text{closure}(\tilde{B})) \geq c_n 2^{n\tau(q)} \nu(\tilde{Q})^q, \quad (43)$$

where $\text{closure}(\tilde{B})$ denotes the closure of \tilde{B} . Moreover,

$$\nu_q\left(B\left(x, 16\sqrt{d}r\right)\right) \geq h(r)r^{-\tau(q)}\nu(B(x, r)), \quad x \in \mathbb{R}^d, 0 < r < \frac{1}{4}, \quad (44)$$

where $h(r) = 6^{-d}2^{-dq}4^{-|\tau(q)|} \inf\{c_n : n \leq \log_2(r^{-1})\}$.

Proof. The proof of the proposition is much similar to that of Proposition 4.3. Here we only point out the essential different point.

Fix an $\epsilon > 0$ and let the measures $(\mu_{\epsilon, s})$, $s < \tau(q)$ be defined identically to that in the proof of Proposition 4.3. Instead of the estimates (38) and (39), by using $b_{\epsilon, n+m} \geq b_{\epsilon, m}$ and (40), we have

$$\begin{aligned} \sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \mu_{\epsilon, s}(\tilde{B}) &\geq \sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \left(Z_\epsilon(s)^{-1} \sum_{m=1}^{\infty} b_{\epsilon, n+m} 2^{(n+m)s} \sum_{Q \in \mathbf{D}_{n+m}: Q \subset \tilde{B}} \nu(Q)^q \right) \\ &\geq c_n 2^{ns} \nu(\tilde{Q})^q. \end{aligned}$$

Let μ_ϵ be a limit point of $(\mu_{\epsilon, s})$ in the weak-star topology as $s \uparrow \tau(q)$. Then

$$\sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \mu_\epsilon(\text{closure}(\tilde{B})) \geq \liminf_{s \uparrow \tau(q)} \sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \mu_{\epsilon, s}(\tilde{B}) \geq c_n 2^{n\tau(q)} \nu(\tilde{Q})^q.$$

This finishes the proof of (43) by letting $\nu_q = \mu_\epsilon$.

Now we prove (44). Let $x \in \mathbb{R}^d$ and $0 < r < 1/4$. Take $n \in \mathbb{N}$ such that $2r < 2^{-n} \leq 4r$. Then

$$B(x, r) \subset \bigcup_{\tilde{Q} \in \mathbf{D}_n, \tilde{Q} \sim B(x, r)} \tilde{Q}.$$

Since there are at most 2^d different $\tilde{Q} \in \mathbf{D}_n$ such that $\tilde{Q} \sim B(x, r)$, by (14) we have

$$\nu(B(x, r))^q \leq \left(\nu \left(\bigcup_{\tilde{Q} \in \mathbf{D}_n, \tilde{Q} \sim B(x, r)} \tilde{Q} \right) \right)^q \leq 2^{dq} \sum_{\tilde{Q} \in \mathbf{D}_n, \tilde{Q} \sim B(x, r)} \nu(\tilde{Q})^q. \quad (45)$$

However for each $\tilde{Q} \in \mathbf{D}_n$ with $\tilde{Q} \sim B(x, r)$, by (43), we have

$$\nu(\tilde{Q})^q \leq \left(c_n 2^{n\tau(q)} \right)^{-1} \sum_{\tilde{B} \in \mathbf{D}_n, \tilde{B} \sim \tilde{Q}} \nu_q(\text{closure}(\tilde{B})).$$

Observe that there are at most 3^d many $\tilde{B} \in \mathbf{D}_n$ so that $\tilde{B} \sim \tilde{Q}$, and for each such \tilde{B} , $\tilde{B} \subset B(x, 16\sqrt{d}r)$. Hence we have

$$\nu(\tilde{Q})^q \leq \left(c_n 2^{n\tau(q)} \right)^{-1} 3^d \nu_q \left(B \left(x, 16\sqrt{d}r \right) \right),$$

Combining it with (45) yields

$$\nu(B(x, r))^q \leq \left(c_n 2^{n\tau(q)} \right)^{-1} 2^{dq} 6^d \nu_q \left(B \left(x, 16\sqrt{d}r \right) \right),$$

that is,

$$\nu_q \left(B \left(x, 16\sqrt{d}r \right) \right) \geq c_n 2^{n\tau(q)} 2^{-dq} 6^{-d} \nu(B(x, r))^q.$$

This proves (44) by observing that $2^{n\tau(q)} = r^{-\tau(q)} (2^n r)^{\tau(q)} \geq r^{-\tau(q)} 4^{-|\tau(q)|}$. \square

As a direct corollary of Proposition 3.1, Proposition 4.3 and Proposition 4.4, we have

COROLLARY 4.5. *Let ν be any self-conformal measure on \mathbb{R}^d . Then the limit $\tau(\nu, q)$ in (2) exists for all $q > 0$.*

THEOREM 4.6. *Let ν be a compactly supported probability measure on \mathbb{R}^d . Assume that the condition of Proposition 4.3 is satisfied for ν for all q in an interval $(a, b) \subset \mathbb{R}$ with $a > 0$. Then for any $\alpha = \tau'(\nu, t)$ with $t \in (a, b)$,*

$$\dim_H E_\nu(\alpha) = \alpha t - \tau(\nu, t) = \inf \{ \alpha q - \tau(\nu, q) : q \in \mathbb{R} \}.$$

Proof. It follows directly from Proposition 4.3 and Proposition 2.1. \square

Proof of Theorem 1.2. It follows directly from Proposition 3.1, Proposition 4.3 and Proposition 4.4. \square

Proof of Theorem 1.1. It follows directly from Proposition 2.1 and the Gibbs property (6) in Theorem 1.2. \square

5. *The asymptotically weak separation condition*

Let $\{S_i\}_{i=1}^\ell$ be a C^1 -conformal IFS on a compact set $X \subset \mathbb{R}^d$, and K the corresponding self-conformal set. For $n \in \mathbb{N}$, let W_n be defined as in (20).

DEFINITION 5.1. The IFS $\{S_i\}_{i=1}^\ell$ is said to satisfy the *asymptotically weak separation condition* (AWSC) if there exists a sequence (t_n) of natural numbers such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log t_n = 0$$

and for each $n \in \mathbb{N}$ and $\tilde{Q} \in \mathbf{D}_n$,

$$\#\{S_u : u \in W_n, K_u \cap \tilde{Q} \neq \emptyset\} \leq t_n. \quad (46)$$

The AWSC is theoretically weaker than the weak separation condition (WSC) introduced by Lau and Ngai [32] in which the (t_n) is asked to be a constant sequence. In the following we give a natural example of IFS which satisfies the AWSC.

DEFINITION 5.2. A real number $\beta > 1$ is said to be a *Pisot number* if it is an algebraic integer whose algebraic conjugates all have modulus less than 1. Whilst $\beta > 1$ is called a *Salem number* if it is an algebraic integer whose algebraic conjugates all have modulus not greater than 1, with at least one of which on the unit circle.

There are infinitely many Pisot numbers and Salem numbers. For example, for each integer $n \geq 2$, the largest positive root of $x^n - x^{n-1} - \dots - x - 1$ is a Pisot number, whilst the largest positive root of $x^{2n} - x^{2n-1} - \dots - x + 1$ is a Salem number. We refer to Salem's book [53] for some interesting properties of Pisot and Salem numbers.

PROPOSITION 5.3. *Let $\beta > 1$ be a Pisot number or Salem number. Then the IFS*

$$\{S_1x = \beta^{-1}x, S_2x = \beta^{-1}x + 1\}$$

on \mathbb{R} satisfies the AWSC.

To prove the above proposition, we need the following standard result about algebraic numbers. For a proof, see [23, Lemma 1.51].

LEMMA 5.4. *Let α be an algebraic integer greater than 1. Let $\alpha_1, \alpha_2, \dots, \alpha_s$ denotes the algebraic conjugates of α and σ denotes the number of i such that $|\alpha_i| = 1$. If $A(x)$ is a polynomial of degree at most n with integer coefficients not exceeding M in modulus for which $A(\alpha) \neq 0$. Then*

$$|A(\alpha)| \geq \frac{\prod_{|\alpha_i| \neq 1} \|\alpha_i - 1\|}{(n+1)^\sigma \left(\prod_{|\alpha_i| > 1} |\alpha_i| \right)^{n+1} M^s}.$$

Proof of Proposition 5.3. If β is a Pisot number, then the IFS satisfies the WSC (see [32, Example 2]), and thus it satisfies the AWSC. In the following we assume that β is a Salem number. Let $\beta_1, \beta_2, \dots, \beta_s$ be the algebraic conjugates of β . Then by Lemma 5.4, if $A(x)$ is a polynomial of degree at most n with integer coefficients not exceeding M in modulus for which $A(\beta) \neq 0$, then

$$|A(\beta)| \geq c(n+1)^{-s} M^{-s}, \quad (47)$$

where $c = \prod_{|\beta_i| \neq 1} \|\beta_i - 1\|$. A direct calculation shows that for any $k \in \mathbb{N}$ and $u = u_1 \dots u_k, v = v_1 \dots v_k \in \{1, 2\}^k$,

$$S_u(0) - S_v(0) = \beta^{-(k-1)} \sum_{i=1}^k (u_i - v_i) \beta^{k-i}.$$

Thus by (47), if $S_u \neq S_v$ then

$$|S_u(0) - S_v(0)| \geq c\beta^{-(k-1)} k^{-s}. \quad (48)$$

Let K be the self-similar set generated by the IFS $\{S_1, S_2\}$. It is clear that $K \subseteq [0, \frac{\beta}{\beta-1}]$ and $\text{diam}(K) = \frac{\beta}{\beta-1}$. Now let $n \in \mathbb{N}$. Then for any $u = u_1 \dots u_k \in W_n$, k is the unique integer such that

$$\frac{\beta}{\beta-1} \cdot \beta^{-k} \leq 2^{-n} < \frac{\beta}{\beta-1} \cdot \beta^{-k+1}.$$

It follows that $\beta^k \leq 2^n \frac{\beta^2}{\beta-1}$ and thus $k \leq n \log_\beta 2 + 2 - \log_\beta(\beta-1)$. Moreover for any two $u, v \in W_n$ with $S_u \neq S_v$, by (48) we obtain

$$|S_u(0) - S_v(0)| \geq c\beta^{-(k-1)} k^{-s} \geq c2^{-n} \cdot \frac{\beta-1}{\beta} \cdot (n \log_\beta 2 + 2 - \log_\beta(\beta-1))^{-s}. \quad (49)$$

Let $\tilde{Q} \in \mathbf{D}_n$. Then for any $u \in W_n$ with $K_u \cap \tilde{Q} \neq \emptyset$, the point $S_u(0)$ has a distance not exceeding 2^{-n} from \tilde{Q} . Hence by (49), we have

$$\begin{aligned} \#\{S_u : u \in W_n, K_u \cap \tilde{Q} \neq \emptyset\} &\leq \frac{3 \cdot 2^{-n}}{c2^{-n} \cdot \frac{\beta-1}{\beta} \cdot (n \log_\beta 2 + 2 - \log_\beta(\beta-1))^{-s}} + 1 \\ &\leq 3c^{-1} \frac{\beta}{\beta-1} \cdot (n \log_\beta 2 + 2 - \log_\beta(\beta-1))^s + 1 \\ &:= c_n, \end{aligned}$$

which deduces the AWSC since $\lim_{n \rightarrow \infty} \frac{\log c_n}{n} = 0$. \square

REMARK 5.5. Proposition 5.3 can be extended slightly. Indeed, by a similar argument one can show that if $\beta > 1$ is a Salem number, then an IFS $\{S_i\}_{i=1}^\ell$ on \mathbb{R} satisfies the AWSC if each S_i has the form

$$S_i x = \pm \beta^{-m_i} x + b_i,$$

where $m_i \in \mathbb{N}$ and $b_i \in \mathbb{Z}[\beta]$. Here $\mathbb{Z}[\beta]$ denotes the integral ring generated by β .

PROPOSITION 5.6. *Let ν be a self-conformal measure on \mathbb{R}^d generated by an IFS $\{S_i\}_{i=1}^\ell$ which satisfies the asymptotically weak separation condition. Then for each $q > 0$, there exists a sequence (c_n) of positive numbers with $\lim_{n \rightarrow \infty} \frac{\log c_n}{n} = 0$ such that for each $n \in \mathbb{N}$ and $\tilde{Q} \in \mathbf{D}_n$,*

$$\sum_{Q \in \mathbf{D}_{m+n}: Q \subset \tilde{Q}} \nu(Q)^q \leq c_n \tau_m(\nu, q) \sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \nu(\tilde{B})^q, \quad (50)$$

and

$$\sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \left(\sum_{Q \in \mathbf{D}_{m+n}: Q \subset \tilde{B}} \nu(Q)^q \right) \geq c_n \tau_m(\nu, q) \nu(\tilde{Q})^q. \quad (51)$$

Proof. Let ν be the self-conformal measure generated by the IFS $\{S_i\}_{i=1}^\ell$ and a probability weight (p_1, \dots, p_ℓ) . For any $n \in \mathbb{N}$, let W_n be defined as in (20). Then it follows from (3) that

$$\nu = \sum_{u \in W_n} p_u \nu \circ S_u^{-1} \quad \text{for all } n \in \mathbb{N}, \quad (52)$$

where $p_u = p_{u_1} \cdots p_{u_k}$ for $u = u_1 \dots u_k$. For each $n \in \mathbb{N}$, we define an equivalence relation “ \approx ” on W_n by setting $u \approx v$ if $S_u = S_v$. For $u \in W_n$, let $[u]$ denote the equivalence class that contains u . In particular, we write

$$p_{[u]} := \sum_{v \in [u]} p_v, \quad S_{[u]} := S_u \quad \text{and} \quad K_{[u]} := K_u.$$

Then (52) can be rewritten as

$$\nu = \sum_{[u] \in W_n / \approx} p_{[u]} \nu \circ S_{[u]}^{-1} \quad \text{for all } n \in \mathbb{N}. \quad (53)$$

Let $q > 0$. By Proposition 3.3, there exists a sequence (d_n) of positive numbers with $\lim_{n \rightarrow \infty} \frac{\log d_n}{n} = 0$, such that for all $m, n \in \mathbb{N}$ and $u \in W_n$,

$$(d_n)^{-1} \tau_m(\nu, q) \leq \sum_{Q \in \mathbf{D}_{m+n}} \nu(S_u^{-1}Q)^q \leq d_n \tau_m(\nu, q). \quad (54)$$

We first prove (50). By (53), for each $\tilde{Q} \in \mathbf{D}_n$ and each $Q \in \mathbf{D}_{m+n}$ with $Q \subset \tilde{Q}$, we have

$$\nu(Q) = \sum_{[u] \in W_n / \approx} p_{[u]} \nu \left(S_{[u]}^{-1}(Q) \right) = \sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} p_{[u]} \nu \left(S_{[u]}^{-1}(Q) \right).$$

Combining it with (14) yields

$$\nu(Q)^q = \left(\sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} p_{[u]} \nu \left(S_{[u]}^{-1}(Q) \right) \right)^q \leq t_n^q \sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} p_{[u]}^q \nu \left(S_{[u]}^{-1}(Q) \right)^q,$$

where (t_n) is the corresponding sequence in Definition 5.1. Summing over $Q \in \mathbf{D}_{m+n}$ with $Q \subset \tilde{Q}$, and using (14) and (54), we obtain

$$\begin{aligned}
\sum_{Q \in \mathbf{D}_{n+m}, Q \subset \tilde{Q}} \nu(Q)^q &\leq \sum_{Q \in \mathbf{D}_{n+m}, Q \subset \tilde{Q}} \left(t_n^q \sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} p_{[u]}^q \nu \left(S_{[u]}^{-1}(Q) \right)^q \right) \\
&\leq t_n^q \sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} \left(\sum_{Q \in \mathbf{D}_{n+m}} p_{[u]}^q \nu \left(S_{[u]}^{-1}(Q) \right)^q \right) \\
&\leq d_n t_n^q \tau_m(\nu, q) \sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} p_{[u]}^q \\
&\leq d_n t_n^{q+1} \tau_m(\nu, q) \left(\sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} p_{[u]} \right)^q. \tag{55}
\end{aligned}$$

Since $\text{diam}(K_u) \leq 2^{-n}$ for $u \in W_n$, $K_u \cap \tilde{Q} \neq \emptyset$ implies $K_u \subset \bigcup_{\tilde{B} \in \mathbf{D}_n, \tilde{B} \sim \tilde{Q}} \tilde{B}$. We have

$$\begin{aligned}
\sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} p_{[u]} &= \sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} p_{[u]} \nu \left(S_{[u]}^{-1} \left(\bigcup_{\tilde{B} \in \mathbf{D}_n, \tilde{B} \sim \tilde{Q}} \tilde{B} \right) \right) \\
&\leq \sum_{[u] \in W_n / \approx} p_{[u]} \nu \left(S_{[u]}^{-1} \left(\bigcup_{\tilde{B} \in \mathbf{D}_n, \tilde{B} \sim \tilde{Q}} \tilde{B} \right) \right) \\
&= \nu \left(\bigcup_{\tilde{B} \in \mathbf{D}_n, \tilde{B} \sim \tilde{Q}} \tilde{B} \right) = \sum_{\tilde{B} \in \mathbf{D}_n, \tilde{B} \sim \tilde{Q}} \nu(\tilde{B}). \tag{56}
\end{aligned}$$

Thus by (14),

$$\left(\sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} p_{[u]} \right)^q \leq \left(\sum_{\tilde{B} \in \mathbf{D}_n, \tilde{B} \sim \tilde{Q}} \nu(\tilde{B}) \right)^q \leq 3^{dq} \sum_{\tilde{B} \in \mathbf{D}_n, \tilde{B} \sim \tilde{Q}} \nu(\tilde{B})^q.$$

Combining this with (55), we have

$$\sum_{Q \in \mathbf{D}_{n+m}, Q \subset \tilde{Q}} \nu(Q)^q \leq 3^{dq} d_n t_n^{q+1} \tau_m(\nu, q) \sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \nu(\tilde{B})^q,$$

which proves (50).

To prove (51), let $\tilde{Q} \in \mathbf{D}_n$. Then for any $Q \in \mathbf{D}_{n+m}$,

$$\nu(Q) = \sum_{[u] \in W_n / \approx} p_{[u]} \nu \left(S_{[u]}^{-1}(Q) \right) \geq \sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} p_{[u]} \nu \left(S_{[u]}^{-1}(Q) \right).$$

It follows from (46) and (14) that

$$\nu(Q)^q \geq t_n^{-1} \sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} p_{[u]}^q \nu \left(S_{[u]}^{-1}(Q) \right)^q. \tag{57}$$

Now let $u \in W_n$ such that $K_u \cap \tilde{Q} \neq \emptyset$. Then $K_u \subset \bigcup_{\tilde{B} \in \mathbf{D}_n, \tilde{B} \sim \tilde{Q}} \tilde{B}$. It implies that for any $Q \in \mathbf{D}_{m+n}$, if $Q \cap K_u \neq \emptyset$ then $Q \subset \bigcup_{\tilde{B} \in \mathbf{D}_n, \tilde{B} \sim \tilde{Q}} \tilde{B}$. Therefore

$$\sum_{Q \in \mathbf{D}_{n+m}} \nu(S_u^{-1}Q)^q = \sum_{Q \in \mathbf{D}_{n+m}, Q \cap K_u \neq \emptyset} \nu(S_u^{-1}Q)^q = \sum_{Q \in \mathbf{D}_{m+n}, Q \subset \bigcup_{\tilde{B} \in \mathbf{D}_n, \tilde{B} \sim \tilde{Q}} \tilde{B}} \nu(S_u^{-1}Q)^q. \quad (58)$$

Thus summing over $Q \in \mathbf{D}_{n+m}$, $Q \subset \bigcup_{\tilde{B} \in \mathbf{D}_n, \tilde{B} \sim \tilde{Q}} \tilde{B}$ in (57) yields

$$\begin{aligned} & \sum_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \left(\sum_{Q \in \mathbf{D}_{m+n}: Q \subset \tilde{B}} \nu(Q)^q \right) \\ &= \sum_{Q \in \mathbf{D}_{m+n}, Q \subset \bigcup_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \tilde{B}} \nu(Q)^q \\ &\geq t_n^{-1} \sum_{Q \in \mathbf{D}_{m+n}, Q \subset \bigcup_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \tilde{B}} \left(\sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} p_{[u]}^q \nu(S_{[u]}^{-1}(Q))^q \right) \\ &= t_n^{-1} \sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} \left(\sum_{Q \in \mathbf{D}_{m+n}, Q \subset \bigcup_{\tilde{B} \in \mathbf{D}_n: \tilde{B} \sim \tilde{Q}} \tilde{B}} p_{[u]}^q \nu(S_{[u]}^{-1}(Q))^q \right) \\ &= t_n^{-1} \sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} \left(\sum_{Q \in \mathbf{D}_{m+n}} p_{[u]}^q \nu(S_{[u]}^{-1}(Q))^q \right) \quad (\text{by (58)}) \\ &\geq d_n^{-1} t_n^{-1} \tau_m(\nu, q) \sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} p_{[u]}^q \quad (\text{by (54)}) \\ &\geq d_n^{-1} t_n^{-(q+1)} \tau_m(\nu, q) \left(\sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} p_{[u]} \right)^q \\ &\geq d_n^{-1} t_n^{-(q+1)} \tau_m(\nu, q) \left(\sum_{[u] \in W_n / \approx, K_{[u]} \cap \tilde{Q} \neq \emptyset} p_{[u]} \nu(S_{[u]}^{-1} \tilde{Q}) \right)^q \\ &= d_n^{-1} t_n^{-(q+1)} \tau_m(\nu, q) \nu(\tilde{Q})^q. \end{aligned}$$

This finishes the proof of (51). \square

The following theorem extends the multifractal result of Lau and Ngai [32] about self-similar measures satisfying the WSC.

THEOREM 5.7. *Let ν be a self-conformal measure on \mathbb{R}^d generated by a C^1 -conformal IFS $\{S_i\}_{i=1}^\ell$ which satisfies the asymptotically weak separation condition. Then for any $\alpha = \tau'(\nu, t)$ with $t > 0$,*

$$\dim_H E_\nu(\alpha) = \alpha t - \tau(\nu, t) = \inf\{\alpha q - \tau(\nu, q) : q \in \mathbb{R}\}. \quad (59)$$

Proof. The theorem follows directly from Proposition 5.6 and Theorem 4.6. \square

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